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Pricing finite maturity game call options with convertible features

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ABSTRACT

This paper considers the pricing problem for game call options on the finite maturity horizon. These financial instruments appear as a natural extension of the American-style options. Their main feature is that the writer can cancel the option prematurely paying some amount above the usual option payment. We assume that this penalty consists of three parts – a proportion of the usual option payment, some shares of the underlying assets, and a fixed amount. We approximate the optimal exercise boundaries for both option's participants by maximizing their financial expectations. This way, a boundary value problem for option pricing arises. We apply the Crank-Nicolson finite difference approach to solve it. Some numerical experiments are performed.

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1. Introduction

The options are one of the most important financial instruments that are traded on the markets. There are many variants but the main are two – European and American. The difference between them is the moment they expire. The European options can be exercised only at the maturity date whereas this can be made at every moment prior to the maturity for the American contracts. Namely, this early exercise right destines the importance of the American-style securities and the major part of the traded options they have. The game options, first considered in Kifer (2000), generalize the concept of the American options adding an early canceling right for the option's writer to the existing such holder's right. To use it, the writer is obliged to pay some penalty above the usual option payment – the traditional assumption is that it is a fixed amount during the option life. The options without maturity constraints are easier for examination due to the lack of the threat of time lapse. We refer to Kyprianou (2004), Suzuki and Sawaki (2007), and Zaevski (2020b) for the put style options, whereas the call versions are examined in Emmerling (2012) and Zaevski (2020a), see also Gapeev, Li, and Wu (2020). Although these perpetual derivatives are not generally tradable at the exchanges, they give very important information for the finite maturity contracts. The call style of such options is examined in Kunita and Seko (2004) and later Yam, Yung, and Zhou (2014) refine some of the derived results. The finite maturity put options are considered in Kühn and Kyprianou (2007) under the assumption that the underlying asset does not pay dividends. Later this restriction is removed in Zaevski (2022). We should also mention several works related to the game-style American options published recently. They are studied in the presence of transaction costs in Roux (2016) and Roux and Zastawniak (2018). Multi-person game derivatives are introduced and examined in Guo and Rutkowski (2017).

The game options as a particular example of the Dynkin games are exposed in Lv, Wu, and Zhang (2022). Options exhibiting Asian features are presented in Guo, Zhang, and Wang (2020). The risk minimization problem for game options is considered in Dolinsky (2020) and Dolinsky and Kifer (2016).

All these papers assume that the penalty that the writer owes is fixed during the option life. Alternatively, in the present paper, we consider a three-component penalty – a proportion of the usual option's payment, some number of shares of the underlying asset, and a fixed amount. Such options without maturity constraints are explored in Zaeveski (2023). We introduce an additional discount factor to establish our model. Along with the time dependence in the payment structures it introduces, this discounting has another outstanding importance. It is proven in Zaeveski (2020a), proposition 2.3, that a model based on a continuously paying dividend underlying asset can be presented as a non-dividend model after a change of variables. This way we may examine a non-dividend model without loss of generality. The undiscounted case is considered separately due to its specifics. It turns out that it is never optimal for the holder to exercise – this fact is well-known for the usual American call options.

Our first task is to determine the shape of the so-called optimal regions – they consist of these points for which the immediate exercise is the best strategy for one of the option's participants. It turns out that the holder's set consists of all points larger than some increasing starting above the strike function (w.r.t. the time to maturity). Namely, this function is the holder's optimal boundary. If the additional discount factor does not exist, then this boundary is infinitely large. The writer's region can be more variable. There exist two important values, possibly infinitely large, for the time to maturity. Before the first one, the writer's optimal region is empty which means that the option is non-cancellable. If the time to maturity is between these values, then only the strike is writer optimal. At last, if the time to maturity is larger than the second critical value, then the writer's optimal region is an interval with the strike for the lower boundary. In such a way, the writer's exercise boundary does not exist for small enough times to maturity, it is the strike for the medium ones, and it increases to the corresponding perpetual value after the second critical value. The points not belonging neither to the writer's optimal region nor to the holder's one form the so-called continuation region – the strategy of keeping the option alive provides better possibilities for both participants. This region consists of two parts – the set between the boundaries and the points below the strike.

The second task is to approximate the optimal boundaries maximizing the financial expectations both for the writer and holder. We do this using some American-style derivatives with a fixed strategy for one of the option's participants and with an early exercise right for the other. A particular class of such instruments is very important – we name them (τ, L) -American options. As usual, the holder has the right to exercise prematurely. In addition, if the asset hits the strike in a moment that provides a time to maturity larger than τ , then the option expires paying the amount of L . Note that if the time to maturity is between the above-mentioned critical values, then the game option turns namely to a (τ, L) -American one, where τ is the first critical value and L is the penalty.

The third task is to derive the fair option price if the immediate exercise is not optimal neither for the holder nor for the writer, i.e. the underlying asset is in the continuation region. As we mentioned above, it consists of two parts. If the initial value is below the strike, we have a one-sided hitting problem. This allows us to derive a semi-closed form formula. The case when the asset starts between the optimal boundaries is somewhat more complicated. But knowing the exercise boundaries, pricing turns into a boundary value problem in a known region. There are many finite difference algorithms for numerical solving of such differential equations. The main of them are the explicit, implicit, and Crank-Nicolson methods. The explicit one is relatively fast, but it may diverge if the ratio between the time step and the square of the price step is larger than 0.5. In addition, it produces an accuracy proportional to the time step and the square of the

price step. The same conclusions hold for the implicit method too but, differently from the explicit one, it converges ever. The price of this is a larger time consumption due to the arising linear system which has to be solved. On the other hand, the accuracy of the Crank-Nicolson finite difference scheme is proportional to the square of the price step as well as the square of the time step. In addition, this algorithm is always numerically stable and convergent. We choose namely the Crank-Nicolson method for our purpose despite it can generate some oscillations in the resulting solution. They may appear since it is A-stable but not L-stable. Nonetheless, we use it since it is unconditionally stable and fast enough. For some alternative numerical methods, we refer to Floc'h (2014). There can be found a discussion on the pros and cons of these algorithms. We present some numerical results to validate the derived theoretical results.

Let us discuss several possible extensions of our model. A possible direction is to generalize the Gaussian assumptions for the log-returns by a Lévy process – see for example Rachev and Mitnik (2000), Boyarchenko and Levendorskiĭ (2002), Bianchi et al. (2010), and Cont and Tankov (2004). Additionally, several authors turn to stochastic volatility to capture some stylized facts such as volatility clustering, leverage effect, etc. We can refer to the classical model of Heston (1993) as well as to a new one of He and Lin (2021a). Some jump extensions can be found in Bates (1996), Zaeviski, Kim, and Fabozzi (2014), and He and Lin (2021b). Let us mention also two other generalizations – He and Chen (2022) introduce regime-switching for the volatility whereas He and Chen (2021) assume that the long-term mean is stochastic. Note that our approach, partly based on the infinitesimal differential operators, can be further applied to the above-mentioned models since they are defined on other Feller-Markov stochastic processes.

The paper is organized as follows. We state our model in Sec. 2. The shape of both optimal regions is obtained in Sec. 3. In Sec. 4, we present our algorithm for the exercise boundaries' approximation. The pricing problem is considered in Sec. 5. The undiscounted case is investigated in Sec. 6. We provide the results of the numerical experiments in Sec. 7.

2. Preliminaries

Let B_t be a Brownian motion under the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$ and the underlying asset be driven by the log-normal process

$$S_t = S_0 \exp \left(\left(r - \frac{\sigma^2}{2} \right) t + \sigma B_t \right). \quad (1)$$

The measure $\mathbb{Q}$ is assumed to be risk-neutral. We denote the infinitesimal generator of process (1) by $\mathcal{A}$ –

$$(\mathcal{A}f)(x) = rxf'(x) + \frac{\sigma^2 x^2}{2} f''(x). \quad (2)$$

We assume that the risk-free and discount rates are the constants r and λ , respectively. The necessary requirements are positiveness for the additional discount rate as well as for the total discount rate – $\lambda \geq 0$ and $r + \lambda > 0$. Note that the risk-free rate can be negative. Let the option maturates at the moment $T < \infty$ and the strike price be K . We assume that the penalty that the option's writer owes for the early canceling right consists of three parts – a proportion $\eta_1 \geq 1$ of the call payoff, $\eta_2 \geq 0$ shares of the underlying asset, and a fixed amount of $\eta_3 \geq 0$. Thus if we denote by $N_1(t, x)$ and $N_2(t, x)$ the payoffs when the holder or writer exercises the option, then

$$\begin{aligned} N_1(t, x) &= e^{-\lambda t} (x - K)^+ \\ N_2(t, x) &= e^{-\lambda t} \left(\eta_1 (x - K)^+ + \eta_2 x + \eta_3 \right). \end{aligned} \quad (3)$$

Let $\mathcal{T}_{[t,T]}$ be the set of all stopping times with values in the interval $[t, T]$ – in fact they are all possible holder's strategies. The set $\mathcal{T}_{[t,T]} \cup \infty$ forms the writer's strategies – the state ∞ means that the writer never cancels prematurely. The holder and writer strategies ζ^b and ζ^s lead to the value function at the point (t, x)

$$M(t, x; \zeta^b, \zeta^s) = \mathbb{E}^{t,x} \left[e^{-r(\zeta^b-t)} N_1(\zeta^b, S_{\zeta^b}) I_{\zeta^b \leq \zeta^s} + e^{-r(\zeta^s-t)} N_2(\zeta^s, S_{\zeta^s}) I_{\zeta^s < \zeta^b} \right]. \quad (4)$$

Hence, the upper and lower values of this game are

$$V^*(t, x) = \inf_{\zeta^b} \sup_{\zeta^s} M(t, x; \zeta^b, \zeta^s); \quad V_*(t, x) = \sup_{\zeta^s} \inf_{\zeta^b} M(t, x; \zeta^b, \zeta^s). \quad (5)$$

It is proven in Zaeovski (2023) that this game exhibits the Nash equilibrium. Thus the value function is defined as $V(t, x) = V_*(t, x) \equiv V^*(t, x)$. Therefore the optimal regions – Y^b and Y^s – and the optimal strategies – ζ^b and ζ^s – for the holder and writer are

$$Y^b = \{(t, x) : V(t, x) = N_1(t, x)\} \text{ and } Y^s = \{(t, x) : V(t, x) = N_2(t, x)\} \\ \zeta^b = \inf\{t : S_t \in Y^b\} \text{ and } \zeta^s = \inf\{t : S_t \in Y^s\}. \quad (6)$$

The Markovian property allows us to work under the time to maturity assuming that the initial moment is zero – we shall denote it by τ and we shall use a subscript to mark the dependence on it.

3. Exercise regions

We shall prove now several propositions that characterize the optimal regions.

Proposition 3.1. *The following six statements hold.*

1. If $\lambda = 0$, then the holder's exercise region is empty – $Y^b = \emptyset$.
2. If $\eta_3 \geq \eta_1 K$, then $Y^s = \emptyset$ and thus the option turns into an ordinary American call.
3. If $x < K$, then $x \in \bar{Y}_\tau$ for all values of τ .
4. If $x \in Y_\tau^b$ and $y > x$, then $y \in Y_\tau^b$ too.
5. If $x \in Y_\tau^s$ and $K \leq y < x$, then $y \in Y_\tau^s$ too.
6. If the penalty is zero, then the immediate exercise is optimal for one of the participants.

Proof. These relations are proven in Sec. 3.1 of the devoted on the perpetual options paper Zaeovski (2023). However, all of them except the second one are proven for $T \leq \infty$. Hence, we need only to investigate the second statement. Suppose that the point $x \in Y_\tau^s$. Let us consider the writer's strategy to not cancel earlier, i.e. $\zeta^s = \infty$, and notate the corresponding holder's optimal strategy by ζ^b . Using $\lambda \geq 0$, $\zeta^b < \infty$, and the martingality of the discounted underlying asset we derive

$$\begin{aligned} 0 &\geq \eta_1(x - K)^+ + \eta_2 x + \eta_3 - \mathbb{E}^x \left[e^{-(r+\lambda)\zeta^b} (S_{\zeta^b} - K)^+ \right] \\ &= \eta_1(x - K)^+ + \eta_2 x + \eta_3 \\ &\quad - \mathbb{E}^x \left[e^{-(r+\lambda)\zeta^b} (S_{\zeta^b} - K) + e^{-(r+\lambda)\zeta^b} (K - S_{\zeta^b})^+ \right] \\ &\geq \eta_1(x - K)^+ + \eta_2 x + \eta_3 \\ &\quad - \mathbb{E}^x \left[e^{-r\zeta^b} S_{\zeta^b} - e^{-(r+\lambda)\zeta^b} K + e^{-(r+\lambda)\zeta^b} (K - S_{\zeta^b})^+ \right] \\ &= \max((\eta_1 - 1)x + \eta_3 - \eta_1 K, \eta_3 - x) + \eta_2 x + E^x \left[e^{-(r+\lambda)\zeta^b} \min(K, S_{\zeta^b}) \right] > 0. \end{aligned} \quad (7)$$

The last inequality is true since $\eta_1 \geq 1$ and $\eta_3 \geq \eta_1 K$. The contradiction finishes the proof. $\square$

Let $V_{am}(x; \tau)$ be the price of the ordinary American option with the same parameters but without the writer's canceling right. Note that this price function is increasing w.r.t. the time to maturity and it starts from zero if the option is at-the-money. Let us define τ_1 as the lower value for which $V_{am}(K; \tau_1) \geq \eta_2 K + \eta_3$. If always $V_{am}(K; \tau) < \eta_2 K + \eta_3$, then we set $\tau_1 = \infty$. The fourth and fifth statements of [Proposition 3.1](#) show that the optimal regions, if they exist, are of the form $Y_\tau^b = (b(\tau), \infty)$ and $Y_\tau^s = (K, a(\tau))$ for some functions $a(\tau) \leq b(\tau)$ – we name them writer's and holder's optimal boundaries, respectively. We shall see later that the writer's region may be the empty set, the singleton $\{K\}$, or an interval. Next, we prove that the option is ordinary American for the small enough maturities.

Proposition 3.2. *The option is ordinary American if and only if $\tau \leq \tau_1$. Something more, as a consequence $b(0) = \max\left\{\frac{r+\lambda}{\lambda}, 1\right\}K$.*

Proof. If we suppose that the option is ordinary American for some $\tau > \tau_1$, then never canceling is the best writer's strategy. However, this is impossible for the at-the-money option since the immediate canceling gives a better result. Inverse, suppose that $Y_\tau^s \neq \emptyset$ for $\tau \leq \tau_1$. The fifth statement of [proposition 3.1](#) leads to $K \in Y_\tau^s$, which cannot be true since never canceling is a better strategy.

As an immediate consequence we see $b(0) = \max\left\{\frac{r+\lambda}{\lambda}, 1\right\}K$. Note that this is true in the case $\eta_2 = \eta_3 = 0$ too (these values lead to $\tau_1 = 0$), because it can be viewed as a limiting case for options with $\eta_2 + \eta_3 > 0$. $\square$

We need the following lemma to establish the boundaries' behavior.

Lemma 3.3. *The function $V(\tau, x)$ is non-decreasing w.r.t. the time to maturity.*

Proof. The proof is very close to one presented in Kunita and Seko (2004) as [Lemma 3.3](#), which is for the case $\eta_1 = \eta_2 = 0$ and $\eta_3 > 0$, i.e. usual game options. It is based on the fact that for $\tau < \tau_1$ the option is ordinary American and hence its value function is increasing. The same is true when $\eta_2 + \eta_3 > 0$ and thus the current proposition is true. The case $\eta_1 > 1$, $\eta_2 = \eta_3 = 0$ can be viewed again as a limiting case. $\square$

Proposition 3.4. *The functions $a(\tau)$ and $b(\tau)$ are non-decreasing w.r.t. the time to maturity. Something more, the writer's boundary is always strictly less the holder's one, $a(\tau) < b(\tau)$ except in the case $\eta_2 = \eta_3 = 0$ and $r \leq 0$.*

Proof. We have that $a(\tau) + \delta \notin Y_\tau^s$ for an arbitrary ϵ and a small enough δ , and thus [lemma 3.3](#) leads to

$$V(\tau - \epsilon, a(\tau) + \delta) \leq V(\tau, a(\tau) + \delta) < N_2(\tau, a(\tau) + \delta) \leq N_2(\tau - \epsilon, a(\tau) + \delta). \quad (8)$$

Therefore $a(\tau) + \delta \notin Y_{\tau-\epsilon}^s$ and as a consequence $a(\tau) + \delta > a(\tau - \epsilon)$. The limit $\delta \rightarrow 0$ proves that $a(\tau) \geq a(\tau - \epsilon)$. Analogously, the statement $b(\tau) - \delta \notin Y_\tau^b$ leads to

$$V(\tau + \epsilon, b(\tau) - \delta) \geq V(\tau, b(\tau) - \delta) > N_1(\tau, b(\tau) - \delta) \geq N_1(\tau + \epsilon, b(\tau) - \delta). \quad (9)$$

Hence, $b(\tau) - \delta \notin Y_{\tau+\epsilon}^b$ or $b(\tau) - \delta < b(\tau + \epsilon)$, and thus $b(\tau) \leq b(\tau + \epsilon)$.

Let us turn to the second part of the proposition assuming $a(\tau) = b(\tau)$ for some $\tau > 0$. Therefore, if $x = a(\tau) \equiv b(\tau)$, then the point (τ, x) leads to one and the same financial result for both of the writer and holder. Hence, $N_1(\tau, x) = N_2(\tau, x)$ and therefore $\eta_2 = \eta_3 = 0$ and $x = K$. If $r > 0$, then [Proposition 3.2](#) gives $b(\tau) \geq b(0) = \max\left\{\frac{r+\lambda}{\lambda}, 1\right\}K > K$ and thus the equality $a(\tau) = b(\tau)$ is impossible.

Suppose now $\eta_2 = \eta_3 = 0$ and $r \leq 0$. The second statement of Theorem 4.1 from Zaeviski (2020d) says that $a(\infty) = b(\infty) = K$. Hence, $a(\tau) = b(\tau) = K$ for all τ because these functions are non-decreasing. $\square$

As we saw above, the case $\eta_2 = \eta_3 = 0$ and $r \leq 0$ is trivial since it leads to $a(\tau) = b(\tau) = K$ for all τ . The option price should be zero below the strike and $(x - K)$ above it. We exclude this case hereafter.

Let us consider $(\bar{\tau}, L)$ -American options defined in the Introduction – we shall denote the related price function by $V_{(\bar{\tau}, L)}(x; \tau)$. Note that if $\bar{\tau} \geq \tau$, then the option is a pure American. Let τ_2 be the time to maturity at which the writer's exercise boundary detaches from the strike. If this never happens, then we state $\tau_2 = \infty$. Namely, this is the case when $r \leq 0$. Also, note that $\tau_2 \geq \tau_1$. The monotonicity of the function $a(\tau)$ leads to the immediate consequence that if $\tau_1 < \infty$ and $a(\infty) = K$, then $\tau_2 = \infty$. Next we characterize the moment τ_2 .

Proposition 3.5. *Let $\tau > \tau_1$. If $\frac{d}{dx} V_{(\tau_1, \eta_2 K + \eta_3)}(\tau, K^+) \leq \eta_1 + \eta_2$, then $a(\tau) = K$. Otherwise, if $\frac{d}{dx} V_{(\tau_1, \eta_2 K + \eta_3)}(\tau, K^+) > \eta_1 + \eta_2$, then $a(\tau) > K$.*

Proof. Suppose that only the point K is writer's optimal. Therefore, there exists a constant $x > K$ such that the interval (K, x) belongs to the continuation region $\bar{Y}_\tau$ and hence $V_{(\tau_1, \eta_2 K + \eta_3)}(\tau, y) < N_2(0, y)$ for all $y \in (K, x)$. This leads to

$$\begin{aligned} \frac{d}{dx} V_{(\tau_1, \eta_2 K + \eta_3)}(\tau, K^+) &= \lim_{y \rightarrow K^+} \frac{V_{(\tau_1, \eta_2 K + \eta_3)}(\tau, y) - V_{(\tau_1, \eta_2 K + \eta_3)}(\tau, K)}{y - K} \\ &\leq \lim_{y \rightarrow K^+} \frac{N_2(0, y) - N_2(0, K)}{y - K} \\ &= \frac{\partial N_2(0, K^+)}{\partial x} = \eta_1 + \eta_2. \end{aligned} \quad (10)$$

Suppose now $a(\tau) > K$. Hence, $V(\tau, y) = N_2(0, y)$ for all small enough but larger than the strike y 's. Note that $V_{(\tau_1, \eta_2 K + \eta_3)}(\tau, y) \geq V(\tau, y)$. Similar arguments as in (10) lead to the desired result. $\square$

Let us consider $\tau = \tau_1$. At this point, the $(\tau_1, \eta_2 K + \eta_3)$ -American option turns to an ordinary American one. The smooth fit principle for American calls leads to $\eta_1 + \eta_2 > 1 = \frac{\partial V_{am}}{\partial x}(\tau_1, b(\tau_1)) > \frac{\partial V_{am}}{\partial x}(\tau_1, K^+)$. The last inequality is true because $V_{am}(\tau, x)$ increases w.r.t. the variable x – see [proposition 3.5](#) from Jalliet, Lamberton, and Lapeyre (1990). Hence $a(\tau_1) = K$ and it stays at the strike for a while now.

Let us denote by $\bar{\eta}$ the optimal boundary of an ordinary American perpetual option with the same parameters. [Theorem 6.1](#) from Zaeviski (2021b) shows that

$$\bar{\eta} = K \frac{(\gamma - 1)^{\gamma-1}}{\gamma^\gamma}, \quad (11)$$

where

$$\gamma = \sqrt{\left(\frac{r}{\sigma^2} - \frac{1}{2}\right)^2 + 2\frac{r + \lambda}{\sigma^2}} - \left(\frac{r}{\sigma^2} - \frac{1}{2}\right). \quad (12)$$

Also, let $a(\infty)$ be the writer's boundary of a perpetual game option with the same parameters which can be derived via Zaeviski (2023). The theorem below formulates the derived results for the optimal boundaries and the related regions.

Theorem 3.6. *The holder's boundary is an increasing function w.r.t. the time to maturity, starting at the value $\max\left(\frac{r+\lambda}{\lambda}, 1\right)K$. The following statements hold for the writer's boundary and the corresponding optimal region.*

1. If $\tau < \tau_1$, then the boundary does not exist and $Y_\tau^s \equiv \emptyset$.
2. If $\tau_1 \leq \tau \leq \tau_2$, then $a(\tau) = K$ and $Y_\tau^s \equiv \{K\}$.
3. If $\tau > \tau_2$, then $a(\tau)$ is an increasing function, $a(\tau) > K$, and $Y_\tau^s \equiv [K, a(\tau))$.
4. If $\bar{\eta} \leq \eta_2 K + \eta_3$, then $\tau_1 = \tau_2 = \infty$.
5. If $\bar{\eta} > \eta_2 K + \eta_3$ and $a(\infty) = K$, then $\tau_1 < \infty$, but $\tau_2 = \infty$.
6. If $\bar{\eta} > \eta_2 K + \eta_3$ and $a(\infty)A > K$, then $\tau_1 \leq \tau_2 < \infty$.

4. Exercise boundaries

We shall present an algorithm to approximate the exercise boundaries. We assume that $\bar{\eta} > \eta_2 K + \eta_3$, because in the opposite case, the option is ordinary American. Suppose that $\lambda > 0$. The undiscounted case is considered later.

First, we derive numerically the value of τ_1 . This is possible since the American option prices can be derived with arbitrary precision. Also, we can derive the values $a(\infty)$ and $b(\infty)$ using Zaevski (2023). Unfortunately, we can not find the value τ_2 through Proposition 3.5 because it depends on the derivative of an American-style option. We can say only when $\tau_2 < \infty$ – namely when $a(\infty) > K$. In such a way our approximation of τ_2 shall be the lowest value of τ for which our algorithm produces a writer's boundary above the strike. Let the time interval be relatively uniform divided into $n \geq 2$ sub-intervals $0 = t_0 < t_1 < \dots < t_n = \tau$. We require the point τ_1 to be a node and the sub-intervals before and after τ_1 to have equal lengths. Let τ_1 be the p -th interval, i.e. $\tau_1 = t_p$. We shall approximate the boundaries by exponents of piece-wise linear functions

$$\begin{aligned} a(t) &= \sum_{i=1}^p \exp(a_i(t)) I_{t \in (t_{i-1}, t_i]} \equiv \sum_{i=1}^p \exp(a_{1,i}t + a_{2,i}) I_{t \in (t_{i-1}, t_i]} \\ b(t) &= \sum_{i=1}^n \exp(b_i(t)) I_{t \in (t_{i-1}, t_i]} \equiv \sum_{i=1}^p \exp(b_{1,i}t + b_{2,i}) I_{t \in (t_{i-1}, t_i]}, \end{aligned} \quad (13)$$

We have $b(t) > a(t) \geq K$. We impose the continuity – $A_{i-1} := a_i(t_i) = a_{i-1}(t_i)$, $i = 1, \dots, p$ and $B_{i-1} := b_i(t_i) = b_{i-1}(t_i)$ for $i = 1, \dots, n$. Assume that the underlying asset starts at a point $x \in (A_0, B_0)$, ζ_1 and ζ_2 are the first hitting moments to the lower and upper boundaries, respectively, and $\zeta = \zeta_1 \wedge \zeta_2$. Since the underlying asset is defined as log-normal process (1), the stopping times ζ_1 and ζ_2 can be viewed as the first hitting times of the Brownian motion to the functions

$$\begin{aligned} c(t) &= \sum_{i=1}^p c_i(t) I_{t \in (t_{i-1}, t_i]} \equiv \sum_{i=1}^p (c_{1,i}t + c_{2,i}) I_{t \in (t_{i-1}, t_i]} \\ d(t) &= \sum_{i=1}^n d_i(t) I_{t \in (t_{i-1}, t_i]} \equiv \sum_{i=1}^n (d_{1,i}t + d_{2,i}) I_{t \in (t_{i-1}, t_i]} \\ c_{1,i} &= \frac{a_{1,i} - r}{\sigma} + \frac{\sigma}{2}, \quad i = 1, \dots, p \\ c_{2,i} &= \frac{a_{2,i} - \ln(x)}{\sigma}, \quad i = 1, \dots, p \\ d_{1,i} &= \frac{b_{1,i} - r}{\sigma} + \frac{\sigma}{2}, \quad i = 1, \dots, n \\ d_{2,i} &= \frac{b_{2,i} - \ln(x)}{\sigma}, \quad i = 1, \dots, n. \end{aligned} \quad (14)$$

Therefore the price function turns into

$$\begin{aligned}
G(x, T; a(t), b(t)) &= \mathbb{E}^x \left[e^{-(r+\lambda)(\zeta_2 - T)} (S_{\zeta_2 - T} - K)^+ I_{(\zeta_2 - T) \leq \zeta_1} \right] \\
&\quad + \mathbb{E}^x \left[e^{-(r+\lambda)\zeta_1} (\eta_1 (S_{\zeta_1} - K)^+ + \eta_2 S_{\zeta_1} + \eta_3) I_{\zeta_1 < (\zeta_2 - T)} \right] \\
&= x \sum_{i=1}^n e^{\sigma d_{2,i}} \mathbb{E} \left[e^{-\psi_{2,i} \zeta_2} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta_2} \right] - K \sum_{i=1}^n \mathbb{E} \left[e^{-(r+\lambda)\zeta_2} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta_2} \right] \\
&\quad + (\eta_1 + \eta_2) x \sum_{i=1}^p e^{\sigma c_{2,i}} \mathbb{E} \left[e^{-\psi_{1,i} \zeta_1} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta_1} \right] \\
&\quad - (\eta_1 K - \eta_3) \sum_{i=1}^p \mathbb{E} \left[e^{-(r+\lambda)\zeta_1} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta_1} \right] \\
&\quad + x e^{-\psi_3 T} \mathbb{E} \left[e^{\sigma B_T} I_{B_T < k, T \leq \zeta} \right] - K e^{-(r+\lambda)T} \mathbb{Q}(B_T < k, T \leq \zeta),
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
\psi_{1,i} &= (r + \lambda) - \left(r - \frac{\sigma^2}{2} \right) - \sigma c_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma c_{1,i} \\
\psi_{2,i} &= (r + \lambda) - \left(r - \frac{\sigma^2}{2} \right) - \sigma d_{1,i} = \lambda + \frac{\sigma^2}{2} - \sigma d_{1,i} \\
\psi_3 &= \lambda + \frac{\sigma^2}{2} \\
k &= \frac{1}{\sigma} \ln \left(\frac{K}{x} \right) - \left(\frac{r}{\sigma} - \frac{\sigma}{2} \right) T.
\end{aligned} \tag{16}$$

If $a(\infty) = K$, i.e. we have a $(\tau_1, \eta_2 K + \eta_3)$ -American call, then price function (15) turns into

$$\begin{aligned}
G(x, T; a(t), b(t) \equiv K) &= \mathbb{E}^x \left[e^{-(r+\lambda)(\zeta_2 - T)} (S_{\zeta_2 - T} - K)^+ I_{(\zeta_2 - T) \leq \zeta_1} \right] \\
&\quad + (\eta_2 K + \eta_3) \mathbb{E}^x \left[e^{-(r+\lambda)\zeta_1} I_{\zeta_1 < (\zeta_2 - T)} \right] \\
&= x \sum_{i=1}^n e^{\sigma d_{2,i}} \mathbb{E} \left[e^{-\psi_{2,i} \zeta_2} I_{\zeta < T, \zeta = \zeta_2} \right] - K \sum_{i=1}^n \mathbb{E} \left[e^{-(r+\lambda)\zeta_2} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta_2} \right] \\
&\quad + (\eta_2 K + \eta_3) \sum_{i=1}^p \mathbb{E} \left[e^{-(r+\lambda)\zeta_1} I_{\zeta \in (t_{i-1}, t_i], \zeta = \zeta_1} \right] \\
&\quad + x e^{-\psi_3 T} \mathbb{E} \left[e^{\sigma B_T} I_{B_T < k, T \leq \zeta} \right] - K e^{-(r+\lambda)T} \mathbb{Q}(B_T < k, T \leq \zeta).
\end{aligned} \tag{17}$$

We shall use also the notation $G(x, T; A_0, \dots, A_p; B_0, \dots, B_n)$ for the option prices. We calculate the six terms in formulas (15) or (17) using [propositions A.6 and A.7](#) from [Appendix A](#). The writer's boundary does not exist before the first critical value, whereas the points of the holder's boundary can be derived by the algorithm presented in [Zaevski \(2021b\)](#) for the ordinary American options. These points are with indexes higher than p . The writer's boundary at τ_1 is equal to the strike. We shall examine now separately the cases $a(\infty) = K$ and $a(\infty) > K$.

4.1. Perpetual writer's boundary equal to the strike

Suppose that $a(\infty) = K$ and therefore the option is $(\tau_1, \eta_2 K + \eta_3)$ -American. Hence we have to approximate only the holder's boundary. We use the following algorithm working backwards.

1. We derive the values $B_p, B_{p+1}, \dots, B_n$, $m \leq p$, using the method presented in Zaevski (2021b) for the ordinary American options. Note that $A_1 = A_2 = \dots = A_p = K$, whereas the values $A_{p+1}, \dots, A_n$ do not exist.
2. Suppose that we have obtained the values $B_m, B_{m+1}, \dots, B_n$. Let for a fixed $x \geq B_m$, $\beta(x)$ maximizes $G(x, T; K, \dots, K; xe^\beta, B_m, \dots, B_n)$ for $\beta \geq 0$. Our approximation of the boundary B_{m-1} is the lowest value of x for which $\beta(x) = 0$. In fact, this is the lowest initial asset value for which the immediate exercise is holder optimal

4.2. Perpetual writer's boundary above the strike

The next possible case is when the perpetual writer's boundary is larger than the strike. The algorithm now includes the following steps.

1. The values of $B_p, B_{p+1}, \dots, B_n$ are derived in the same way.
2. Suppose that we know the values $A_m, A_{m+1}, \dots, A_p$ and $B_m, B_{m+1}, \dots, B_n$, $m \leq p$.
3. Let for a fixed $x \geq B_m$ and $\beta \geq 0$, the negative value $\alpha(x, \beta)$ minimizes $G(x, T; xe^\alpha, A_m, \dots, A_p; xe^\beta, B_m, \dots, B_n)$. The point $xe^{\alpha(x, \beta)}$ is the writer's optimal strategy if the holder follows a strategy xe^β . Let $\beta(x) \geq 0$ maximizes $G(x, T; xe^{\alpha(x, \beta)}, A_m, \dots, A_p; xe^\beta, B_m, \dots, B_n)$. Our approximation for the holder's boundary B_{m-1} is the lowest x for which $\beta(x) = 0$ – this the lower holder's optimal point.
4. Analogously, for a fixed $x \geq A_m$ and $\alpha \leq 0$ let $\beta(x, \alpha) \geq 0$ maximizes $G(x, T; xe^\alpha, A_m, \dots, A_p; xe^\beta, B_m, \dots, B_n)$ and $\alpha(x) \leq 0$ minimizes $G(x, T; xe^\alpha, A_m, \dots, A_p; xe^{\beta(x, \alpha)}, B_m, \dots, B_n)$. Our approximation for the writer's optimal boundary A_{m-1} is the highest value of x for which $\alpha(x) = 0$ – this is the highest writer's optimal initial point.

5. Pricing

We continue with the assumption $\lambda > 0$. This means that the holder's boundary is finite. Let the time to maturity be fixed at level τ . If $\tau \leq \tau_1$ we have an ordinary American option. Suppose now $\tau > \tau_1$. Since the continuation region consists of two parts – one below the strike and another between the optimal boundaries – we have to examine these cases separately.

5.1. Initial asset value between the boundaries

Once we know the exercise boundaries, we can view option pricing as the following boundary value problem (BVP)

$$\begin{aligned}
 V_t(t, x) + (\mathcal{A}V)(t, x) - rV(t, x) &= 0 \\
 V(t, a(t)) &= N_2(t, a(t)), \quad t \in (0, \tau - \tau_1) \\
 V(t, b(t)) &= N_1(t, b(t)), \quad t \in (0, \tau) \\
 V(T, x) &= N_1(t, x), \quad x \in \left(0, K \max\left\{1, \frac{r + \lambda}{\lambda}\right\}\right).
 \end{aligned} \tag{18}$$

The equation holds in the strip $\{(0, \tau - \tau_1) \times (a(t), b(t))\} \cup \{(\tau - \tau_1, \tau) \times (0, b(t))\}$ and the boundary conditions are imposed on the lower, upper, and the right boundaries. Note that for $t > \tau - \tau_1$ the lower boundary does not exist – it turns to zero. We use the following Crank-Nicolson finite difference approach to solve BVP (18).

1. We examine the set $\{(0, \tau) \times (K, b(\tau))\}$. We devise uniformly the time interval into M sub-intervals and the state space into N sub-intervals. The time division is made backwards, i.e. $\tau = t_1 > t_2 > \dots > t_M = 0$.
2. We calculate the values of the optimal boundaries in $\overline{N} < N$ and $\overline{M} < M$ points through the algorithm presented in [Sec. 4](#). We set the values for the lower boundary after the critical value τ_1 to be equal to the strike.
3. We use a cubic spline interpolation to derive the boundaries at all M time-points. We denote them by $A_1, A_2, \dots, A_m$ and $B_1, B_2, \dots, B_m$.
4. We set the price function at the maturity to $V(1, n) = N_1(\tau, x_n)$.
5. *The lower boundary.* For every $m \geq 2$ we derive the lower value n such that $x_n \leq A_n$. We denote it by l_m . Note that if $A_m = K$, then $l_m = 1$. If $n \leq l_m$ and $t_m < \tau - \tau_1$, we set $V(m, n) = N_2(t_m, x_n)$. If $t_m \geq \tau - \tau_1$, then $l_m = 1$ and the lower boundary does not exist. We derive the value of $V(m, 1)$ as the price of an ordinary American option using Zaeovski (2021b).
6. *The upper boundary.* For every $m \geq 2$, we find the higher value of n such that $x_n \geq A_n$. We denote it by L_m . We set $V(m, n) = N_1(t_m, x_n)$ for $n \geq L_m$.
7. Once we have derived the values $V(i, n)$ for all n and $i < m$, we can use the Crank-Nicolson finite difference method to find the values $V(m, n)$, $l_m + 1 \leq n \leq L_m - 1$. Eq. (B15) from [Appendix B](#) approximate the derivatives $V_t(t, x)$, $V_x(t, x)$, and $V_{xx}(t, x)$. In such a way the differential equation from BVP (18) turns into (B16). Rearranging these equations for $n \in \{l_m + 1, l_m + 2, \dots, L_m - 1\}$ we derive a linear system for $V(m, n)$ consisting of Eqs. (B17) – (B19). We discuss this system at the end of [Appendix B](#).

5.2. Initial asset price below the strike

Suppose now that the underlying asset starts at a point $x < K$ and $\tau > \tau_1$. The option expires if the asset hits the strike before the moment $\tau^* = \tau - \tau_1$ – then the holder exercises. If this not happen, then the option turns into a classical American. We shall use the following notations:

$$a_{1,2} = \frac{r}{\sigma} \pm \frac{\sigma}{2}$$

$$\mu(y, z) = \frac{\ln y - \ln z}{\sigma}. \quad (19)$$

The option price can be derived as

$$\begin{aligned}
V(x) &= \mathbb{E} \left[e^{-(r+\lambda)\zeta} (\eta_2 K + \eta_3) I_{\zeta \leq \tau^*} \right] \\
&\quad + e^{-r\tau^*} \int_{-\infty}^{-a_2\tau^* + \mu(K, x)} e^{-\lambda\tau^*} V_{am} \left(\tau_1, x e^{\left(r - \frac{\sigma^2}{2}\right)\tau^* + \sigma y} \right) d\mathbb{Q}(B_{\tau^*} < y, \zeta > \tau^*) \\
&= (\eta_2 K + \eta_3) L(\tau^*, r + \lambda, b_1, b_2) \\
&\quad + e^{-(r+\lambda)\tau^*} \int_{-\infty}^{-a_2\tau^* + \mu(K, x)} V_{am} \left(\tau_1, x e^{\left(r - \frac{\sigma^2}{2}\right)\tau^* + \sigma y} \right) f(y; \tau^*) dy,
\end{aligned} \quad (20)$$

where the functions $f(\cdot)$ and $L(\cdot)$ are given by [Eqs. \(A5\) and \(A6\)](#), respectively. Changing the variables as

$$z = x e^{\left(r - \frac{\sigma^2}{2}\right)\tau^* + \sigma y} \iff y = -a_2\tau^* + \mu(z, x) \quad (21)$$

and having in mind $dy = \frac{dz}{\sigma z}$ we transform the price (20) into

$$\begin{aligned} V(x) = & (\eta_2 K + \eta_3) L(\tau^*, r + \lambda, -a_2, \mu(K, x)) \\ & + e^{-(r+\lambda)\tau^*} \int_0^K V_{am}(\tau_1, z) g(z; \tau^*, x) dz, \end{aligned} \quad (22)$$

where the function $g(\cdot)$ is defined as

$$g(z; t, x) = \frac{1}{\sigma z \sqrt{2\pi t}} \left(1 - \exp \left(-\frac{2\mu(K, x)\mu(K, z)}{t} \right) \right) \exp \left(-\frac{(\mu(z, x) - a_2 t)^2}{2t} \right). \quad (23)$$

6. Undiscounted case

Suppose now that $\lambda = 0$. Let $n(\cdot)$ and $N(\cdot)$ be the probability density and cumulative distribution functions of the standard normal distribution. Having in mind [Proposition 3.2](#) we see that $b(0) = \infty$ and therefore $b(\tau) = \infty$ for every τ . This means that the early exercise is never holder's optimal and thus the related American option turns into a European one. Thus we need to modify the derived results. The critical values are obtained *via* the following theorem.

Theorem 6.1. *The first critical value τ_1 is finite if and only if $\eta_3 < K(1 - \eta_2)$. Let the constant M be defined as*

$$M = \frac{2r(\eta_1 K - \eta_3)}{(\eta_1 + \eta_2 - 1)(2r + \sigma^2)}. \quad (24)$$

The second critical value τ_2 is finite if and only if $M > K$.

Proof. The first critical value τ_1 has to be now this time to maturity for which the price of the European at-the-money option is equal to $\eta_2 K + \eta_3$. Having in mind that this price is an increasing from 0 to K function, we see that $\tau_1 < \infty$ when $\eta_3 < K(1 - \eta_2)$.

The statement for the second critical value τ_2 follows the fact that $\max\{K, M\}$ is just the writer's perpetual optimal boundary – see Sec. 3.4 from Zaeviski (2023). $\square$

The next three propositions characterize both critical values τ_1 and τ_2 as well as the value of a $(\bar{\tau}, L)$ -derivative.

Proposition 6.2. *Let the constants a_1 and a_2 be defined as in (19). If τ_1 is finite, then it is the solution of the following equation w.r.t. the variable τ :*

$$\eta_2 K + \eta_3 = K [N(a_1 \sqrt{\tau}) - e^{-r\tau} N(a_2 \sqrt{\tau})]. \quad (25)$$

Proof. The proposition is true, because the right hand-side of [Eq. \(25\)](#) is just the price of the European at-the-money option. $\square$

Proposition 6.3. *Let $\tau > \bar{\tau}$, $\tau^* = \tau - \bar{\tau}$, the underlying asset starts at the value $S_0 = x$, and the function $\mu(\cdot, \cdot)$ be defined as in (19). The price of the $(\bar{\tau}, L)$ -derivative is*

$$\begin{aligned} V_{(\bar{\tau}, L)}(x, \tau) = & L \left[\frac{x}{K} N \left(-a_1 \sqrt{\tau^*} + \frac{\mu(K, x)}{\sqrt{\tau^*}} \right) + \left(\frac{x}{K} \right)^{-\frac{2\tau^*}{\sigma^2}} N \left(a_1 \sqrt{\tau^*} + \frac{\mu(K, x)}{\sqrt{\tau^*}} \right) \right] \\ & + e^{-r\tau^*} \int_K^\infty V_{eu}(z, \bar{\tau}) g(z; \tau^*, x) dz, \end{aligned} \quad (26)$$

where $V_{eu}(\cdot, \cdot)$ is the Black-Scholes pricing formula for the European call option and the function $g(\cdot)$ is given by formula (23).

Proof. Let the initial value x be fixed and ζ be the first hit of the underlying asset to the strike or equivalently, the first hit of the Brownian motion to the linear function $-a_2t + \mu(K, x)$. The price of the $(\bar{\tau}, L)$ -derivative can be obtained as

$$V_{(\bar{\tau}, L)}(x, \tau) = L\mathbb{E}^x \left[e^{-r\zeta} I_{\zeta \leq \tau^*} \right] + e^{-r\tau^*} \int_{-a_2\tau^* + \mu(K, x)}^{\infty} V_{eu} \left(xe^{(r - \frac{\sigma^2}{2})\tau^* + \sigma y}, \bar{\tau} \right) d\mathbb{Q}(B_{\tau^*} < y, \zeta > \tau^*). \quad (27)$$

We finish the proof similarly to [Sec. 5.2](#) – note that we have now a lower hitting problem and thus we use the symmetry of the Brownian motion. $\square$

Proposition 6.4. Let $M > K$ or equivalently $\tau_2 < \infty$ – see [Theorem 6.1](#). If τ_1 is the first critical value, then the second one can be presented as $\tau_2 = \tau_1 + t$, where t satisfies

$$\begin{aligned} & \frac{\eta_2 K + \eta_3}{K} \left[1 - \left(1 + 2 \frac{r}{\sigma^2} \right) N(a_1 \sqrt{t}) - \frac{2}{\sigma \sqrt{t}} n(a_1 \sqrt{t}) \right] \\ & + \frac{2e^{-rt}}{K\sigma^2 t^{\frac{3}{2}}} \int_K^{\infty} \frac{V_{eu}(z, t) \mu(z, K) n\left(a_2 \sqrt{t} - \frac{\mu(z, K)}{\sqrt{t}}\right)}{z} dz = \eta_1 + \eta_2. \end{aligned} \quad (28)$$

Proof. [Proposition 3.5](#) shows that the holder's exercise boundary detaches from the strike when $\frac{d}{dx} V_{(\tau_1, \eta_2 K + \eta_3)}(\tau, K^+) = \eta_1 + \eta_2$. We obtain the x -derivative of function (26) at the point $x = K$ using $\mu(K, K) = 0$, $\mu(y, z) = -\mu(z, y)$ and the following form of the derivative $g_x(z; t, x)$

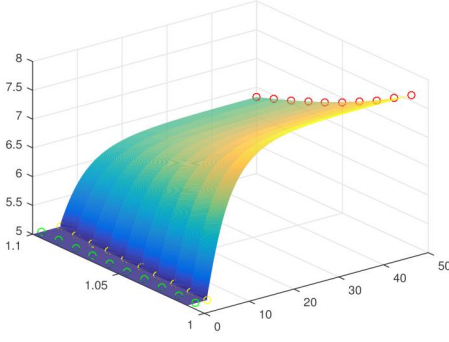
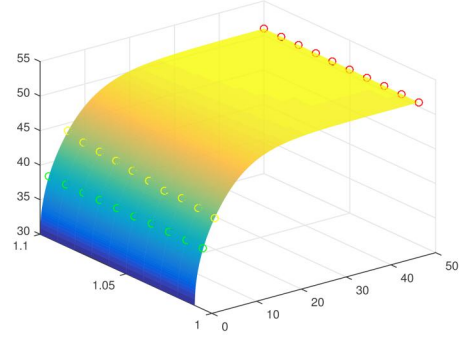
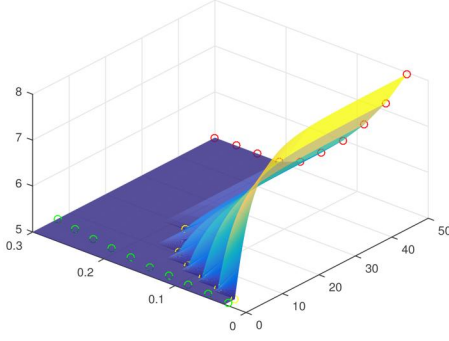
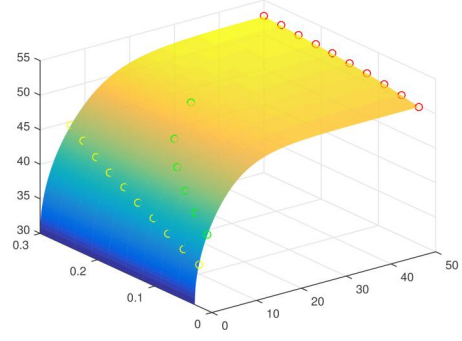
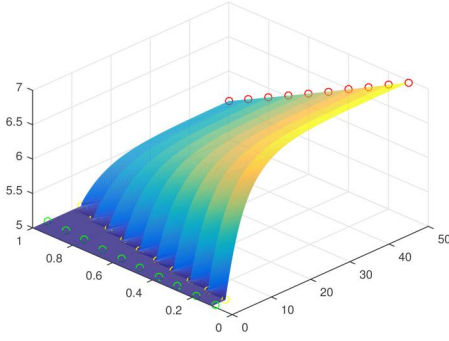
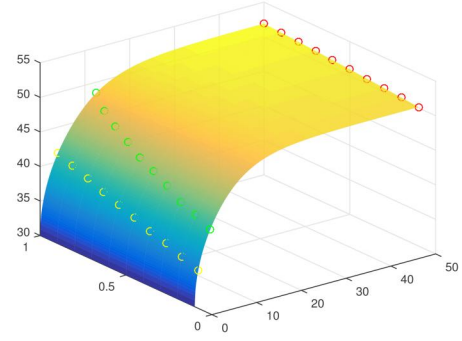
$$g_x(z; t, x) = \frac{\exp\left(-\frac{(\mu(z, x) - a_2 t)^2}{2t}\right)}{t\sigma^2 x z \sqrt{2\pi t}} \left[\begin{aligned} & -2\mu(K, z) \exp\left(-\frac{2\mu(K, x)\mu(K, z)}{t}\right) \\ & + \left(1 - \exp\left(-\frac{2\mu(K, x)\mu(K, z)}{t}\right)\right) (\mu(z, x) - a_2 t) \end{aligned} \right], \quad (29)$$

which leads to

$$g_x(z; t, K) = \frac{2\mu(z, K)}{t\sigma^2 K z \sqrt{2\pi t}} \exp\left(-\frac{(\mu(z, K) - a_2 t)^2}{2t}\right). \quad (30)$$

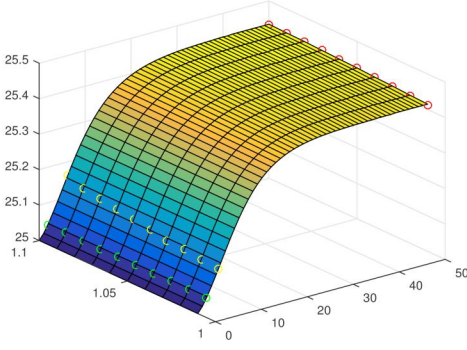
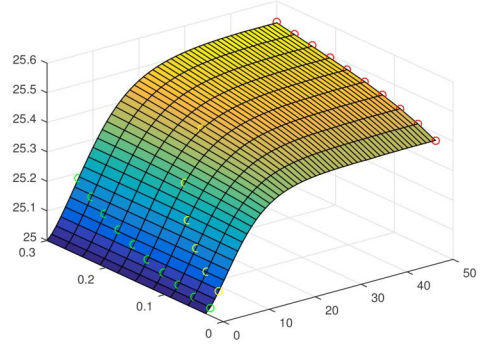
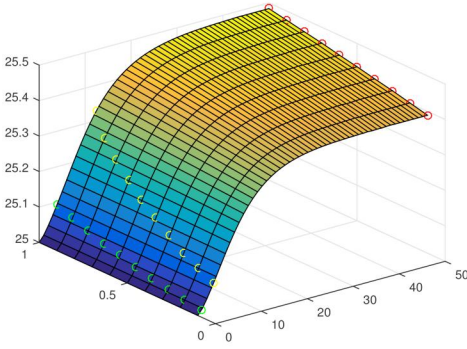
We finish the proof by several easy calculations. $\square$

Let us turn to the pricing problem. If $\tau \leq \tau_1$, then the option is ordinary European. If $\tau_1 < \tau < \tau_2$, then we have a $(\bar{\tau}, L)$ -derivative and we can use [Proposition 6.3](#). It is left to consider the case $\tau > \tau_2$. Note that this is possible only when the strike is below the given by [Eq. \(24\)](#) constant M . We shall first approximate the optimal boundary in the interval $[0, \tau - \tau_2)$. Note that the boundary in the interval $[\tau - \tau_2, \tau - \tau_1)$ is the strike, and it vanishes for $\tau > \tau - \tau_1$. Let the interval $[0, \tau - \tau_2)$ be divided into p -subintervals $0 = t_0 < t_1 < \dots < t_p = \tau - \tau_2$. Let the function $a(t)$ be an exponent of a continuous piecewise linear function at this grid and $A_i = a(t_i)$. Let ζ be the underlying asset's first hit to this function. It can be viewed as the first hit of the Brownian motion to the function $c(t)$ given by formula (14). We can use again some symmetry arguments to [Eqs. \(A14\)](#) and [\(A12\)](#) for $\xi = 0$ together with the second part of [Proposition A.1](#) to derive the cumulative distribution function of the Brownian motion when $\zeta > \tau - \tau_2$. This way the price of


 (a) writer's boundary w.r.t η_1

 (b) holder's boundary w.r.t η_1

 (c) writer's boundary w.r.t. η_2

 (d) holder's boundary w.r.t η_2

 (e) writer's boundary w.r.t. η_3

 (f) holder's boundary w.r.t η_3
Figure 1. Exercise boundaries.

the related derivative can be written as

$$\begin{aligned}
 G(x, T; A_0, \dots, A_p) = & \mathbb{E}^x \left[e^{-r\zeta} (\eta_1 (S_\zeta - K)^+ + \eta_2 S_\zeta + \eta_3) I_{\zeta < \tau - \tau_2} \right] \\
 & + e^{-r(\tau - \tau_2)} \int_K^\infty V_{(\tau_1, \eta_2 K + \eta_3)}(z, \tau_2) \frac{h(-a_2(\tau - \tau_2) + \mu(z, x))}{\sigma z} dz,
 \end{aligned} \tag{31}$$

(a) option prices w.r.t. η_1 (b) option w.r.t. η_2 (c) option w.r.t. η_3 **Figure 2.** Option prices.

where the function $h(y)$ is defined as

$$h(y) = \int_{\alpha_1, \dots, \alpha_{p-1}} \left(\frac{\prod_{i=1}^{p-1} \left(1 - \exp \left(-\frac{2(\alpha_{i-1} - x_{i-1})(\alpha_i - x_i)}{t_i - t_{i-1}} \right) \right)}{\prod_{i=1}^{p-1} \frac{\exp \left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right)}{\sqrt{2\pi(t_i - t_{i-1})}}} \right) dx_1 \dots dx_{p-1} \quad (32)$$

$$f(y, t, a, b) = \frac{1}{\sqrt{2\pi t}} \left(1 - \exp \left(-\frac{2b(at + b - y)}{t} \right) \right) \exp \left(-\frac{y^2}{2t} \right)$$

$$\alpha_i = c_{i,1}t_i + c_{i,2}.$$

The first part of price (31) can be transformed into

$$\begin{aligned} & \mathbb{E}^x \left[e^{-r\zeta} (\eta_1 (S_\zeta - K)^+ + \eta_2 S_\zeta + \eta_3) I_{\zeta < \tau - \tau_2} \right] \\ &= (\eta_1 + \eta_2) x \sum_{i=1}^p e^{\sigma c_{2,i}} \mathbb{E} \left[e^{-\psi_i \zeta} I_{\zeta \in (t_{i-1}, t_i]} \right] - (\eta_1 K - \eta_3) \sum_{i=1}^p \mathbb{E} \left[e^{-r\zeta} I_{\zeta \in (t_{i-1}, t_i]} \right], \end{aligned} \quad (33)$$

where $\psi_i = \frac{\sigma^2}{2} - \sigma c_{1,i}$. The expectations in formula (33) can be derived using the symmetric analogue of formula (A11) assuming that the upper boundary vanishes – this way we have to use formulas (A3) instead (A2). Note that all upper boundaries should be plus infinity.

The next step is to adopt the algorithm from [Sec. 4.2](#) to approximate the writer’s boundary. Supposing that the values $A_m, A_{m+1}, \dots, A_p$ are derived, we define $\alpha(x)$ for a fixed $x \geq A_m$ as the value of $\alpha \leq 0$ which minimizes $G(x, T; xe^\alpha, A_m, \dots, A_p)$. Our approximation for the writer’s optimal boundary A_{m-1} is the highest value of x for which $\alpha(x) = 0$.

Finally, we have to modify BVP (18). The upper boundary has to be removed since the holder’s boundary is infinitely large. Instead this, it is helpful to introduce an auxiliary large enough boundary B . An appropriate approximation of it is the price of the related European option. This way the upper boundary condition in BVP (18) can be taken as $V(t, B) = V_{eu}(t, B)$, $t \in (0, \tau)$.

7. Numerical results

We provide several numerical experiments to confirm our theoretical results. The main parameters we use are – risk-free rate $r = 0.05$, additional discount factor $\lambda = 0.01$, volatility $\sigma = 0.3$, strike price $K = \$5$, penalty proportion of the usual option payment $\eta_1 = 1.1$, $\eta_2 = 0.1$ shares of the underlying asset, and $\eta_3 = \$0.5$ fixed amount. We assume that the initial asset price is $S_0 = \$30$.

In [Figure 1](#) we present the behavior of the optimal boundaries varying the penalty coefficients as $\eta_1 \in (1.01, 1.1)$, $\eta_2 \in (0.03, 0.3)$, and $\eta_3 \in (0.1, 1.1)$. The time to maturity is taken to be between 0 and 50. The green and yellow points present both important time to maturity values. The option is pure American when the time to maturity is less than the green points – waiting the maturity is the best writer’s strategy. If the time to maturity is between the marked points, then the writer cancels the option only when the underlying asset hits the strike. The red points present the perpetual values of the optimal boundaries calculated using the results of [Zaevski \(2023\)](#). We can see that the optimal boundaries tend namely to these values for the

Table 1. Option prices and optimal boundaries.

	writer	holder	$S_0 = 6$	$S_0 = 10$	$S_0 = 15$	$S_0 = 20$	$S_0 = 25$
penalty coefficients $\eta_1 = 1.01, \eta_2 = 0.03, \eta_3 = 0.1$							
$\lambda = 0.01$	9.6870	39.8223	1.2900	5.4476	10.3841	15.2569	20.1483
$\lambda = 0.02$	7.8353	23.2200	1.2900	5.3512	10.1549	15.0261	20.0000
$\lambda = 0.03$	6.8853	17.6892	1.2900	5.2444	10.0342	15.0000	20.0000
$\lambda = 0.04$	6.3098	14.8952	1.2900	5.1595	10.0000	15.0000	20.0000
$\lambda = 0.05$	5.9233	13.1866	1.2896	5.0965	10.0000	15.0000	20.0000
penalty coefficients $\eta_1 = 1.02, \eta_2 = 0.06, \eta_3 = 0.2$							
$\lambda = 0.01$	6.2689	39.8469	1.5800	5.5576	10.3932	15.2581	20.1485
$\lambda = 0.02$	5.6823	23.2740	1.5719	5.4069	10.1611	15.0273	20.0000
$\lambda = 0.03$	5.3090	17.8260	1.5454	5.2865	10.0385	15.0000	20.0000
$\lambda = 0.04$	5.0542	15.1198	1.5150	5.1955	10.0002	15.0000	20.0000
$\lambda = 0.05$	5.0000	13.4834	1.4875	5.1280	10.0000	15.0000	20.0000
penalty coefficients $\eta_1 = 1.03, \eta_2 = 0.09, \eta_3 = 0.3$							
$\lambda = 0.01$	5.0000	39.8515	1.7857	5.5840	10.3961	15.2585	20.1486
$\lambda = 0.02$	5.0000	23.2915	1.7452	5.4324	10.1638	15.0257	20.0000
$\lambda = 0.03$	5.0000	17.8912	1.7075	5.3113	10.0402	15.0000	20.0000
$\lambda = 0.04$	5.0000	15.2440	1.6745	5.2196	10.0005	15.0000	20.0000
$\lambda = 0.05$	5.0000	13.6609	1.6449	5.1507	10.0000	15.0000	20.0000
penalty coefficients $\eta_1 = 1.04, \eta_2 = 0.12, \eta_3 = 0.4$							
$\lambda = 0.01$	5.0000	39.8404	1.9346	5.5977	10.3968	15.2586	20.1486
$\lambda = 0.02$	5.0000	23.2859	1.8921	5.4452	10.1645	15.0258	20.0000
$\lambda = 0.03$	5.0000	17.8979	1.8523	5.3232	10.0412	15.0000	20.0000
$\lambda = 0.04$	5.0000	15.2703	1.8173	5.2306	10.0005	15.0000	20.0000
$\lambda = 0.05$	5.0000	13.7067	1.7856	5.1608	10.0000	15.0000	20.0000
penalty coefficients $\eta_1 = 1.05, \eta_2 = 0.15, \eta_3 = 0.5$							
$\lambda = 0.01$	5.0000	39.7878	2.0408	5.5997	10.3968	15.2586	20.1486
$\lambda = 0.02$	5.0000	23.2472	1.9843	5.4466	10.1645	15.0258	20.0000
$\lambda = 0.03$	5.0000	17.8689	1.9291	5.3241	10.0407	15.0000	20.0000
$\lambda = 0.04$	5.0000	15.2457	1.8792	5.2312	10.0006	15.0000	20.0000
$\lambda = 0.05$	5.0000	13.6823	1.8343	5.1611	10.0000	15.0000	20.0000

large maturities. The option price behavior is presented in Figure 2. The used time and state divisions are with $M = 100$ and $N = 10\,000$ nodes.

Some option prices together with the optimal boundaries are reported in Table 1. The discount factor is amongst $\lambda \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$ and the considered initial asset prices are $S_0 = \$6$, $S_0 = \$10$, $S_0 = \$15$, $S_0 = \$20$, and $S_0 = \$25$. We vary the penalty coefficients in the intervals $\eta_1 \in [1.01, 1.05]$, $\eta_2 \in [0.03, 0.15]$, and $\eta_3 \in [0.1, 0.5]$. The first two columns present the optimal boundaries (the writer's is first and the holder's is second). The rest of the table consists of the corresponding option prices.

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Appendix A. First hitting time propositions

Let the functions $a(\cdot)$ and $b(\cdot)$ be continuous and piecewise linear w.r.t. the time grid $0 \equiv t_0 < t_1 < \dots < t_n \equiv T$.

$$\begin{aligned} a(t) &= (a_{1,m}t + a_{2,m})I_{t_{m-1} < t \leq t_m} \\ b(t) &= (b_{1,m}t + b_{2,m})I_{t_{m-1} < t \leq t_m}. \end{aligned} \tag{A1}$$

Let also $a(0) < 0 < b(0)$, and $a(t) < b(t)$ for all $t \leq T$. It is possible the lower function $a(\cdot)$ to vanish after some t_k , $k \leq n$. We denote the values at the endpoints by $\alpha_i = a(t_i)$ and $\beta_i = b(t_i)$ and let $\gamma_i = \beta_i - \alpha_i > 0$. Let τ_1 and τ_2 be the first hitting moments of a Brownian motion to the functions $a(t)$ and $b(t)$ and τ be the lower one, $\tau = \tau_1 \wedge \tau_2$. We shall use the following notations

$$\begin{aligned}
q_{ij}(\gamma, z) = & \exp\left(-\frac{2[j\gamma_{i-1} + \alpha_{i-1} - \gamma][j\gamma_i + \alpha_i - z]}{t_i - t_{i-1}}\right) \\
& - \exp\left(-\frac{2j[j\gamma_{i-1}\gamma_i + \gamma_{i-1}(\alpha_i - z) - \gamma_i(\alpha_{i-1} - \gamma)]}{t_i - t_{i-1}}\right) \\
& + \exp\left(-\frac{2[j\gamma_{i-1} - (\beta_{i-1} - \gamma)][j\gamma_i - (\beta_i - z)]}{t_i - t_{i-1}}\right) \\
& - \exp\left(-\frac{2j[j\gamma_{i-1}\gamma_i - \gamma_{i-1}(\beta_i - z) + \gamma_i(\beta_{i-1} - \gamma)]}{t_i - t_{i-1}}\right)
\end{aligned} \tag{A2}$$

for $i < k$, $\alpha_{i-1} < \gamma < \beta_{i-1}$, and $\alpha_i < z < \beta_i$. If $i \geq k$, then

$$\begin{aligned}
q_{i,1}(\gamma, z) &= \exp\left(-\frac{2(\alpha_{i-1} - \gamma)(\alpha_i - z)}{t_i - t_{i-1}}\right) \\
q_{ij}(\gamma, z) &= 0, \quad j > 1.
\end{aligned} \tag{A3}$$

We present below several propositions for the Brownian motion's first exit from a strip. The lower versions can be found in Zaeovski (2022) as an appendix.

Proposition A.1. *The probability that the fist hitting moment to the linear function $b(t) = b_1t + b_2$, $b_2 > 0$ is before T is*

$$g(T; b_1, b_2) := \mathbb{P}(\tau_1 < T) = 1 - N\left(\frac{b_1T + b_2}{\sqrt{T}}\right) + \exp(-2b_1b_2)N\left(\frac{b_1T - b_2}{\sqrt{T}}\right). \tag{A4}$$

If this hitting time is after T , then the cumulative distribution function of the Brownian motion is $\mathbb{P}(B_T < \gamma, \tau > T) = \int_{-\infty}^{\gamma} f(u)du$, where

$$f(u; T, b_1, b_2) = \frac{1}{\sqrt{2\pi T}} \left(1 - \exp\left(-\frac{2b_2(b(T) - u)}{T}\right)\right) \exp\left(-\frac{u^2}{2T}\right). \tag{A5}$$

Proof. See Proposition 3.1 and Eq. (3.7) from Zaeovski (2020c). □

Proposition A.2. *Let $\xi > 0$ and the function $b(t) = b_1t + b_2$, $b_2 > 0$, be linear. Then*

$$L(T, \xi; b_1, b_2) := \mathbb{E}[e^{-\xi\tau}I_{\tau \leq T}] = e^{b_2(\sqrt{b_1^2 + 2\xi} - b_1)} g\left(T; \sqrt{b_1^2 + 2\xi}, b_2\right), \tag{A6}$$

where $g(\cdot)$ is the function given by Eq. (A4).

Proof. The proposition is proven in Zaeovski (2020c) as theorem 3.1. □

Proposition A.3. *Suppose that the Brownian motion stays below the linear function $b(t) = b_1t + b_2$, $b_2 > 0$ before the moment T and $z < b(T)$. Then we have*

$$\begin{aligned}
V(\theta, z, T; b_1, b_2) &\equiv \mathbb{E}[e^{\theta B_T} I_{B_T > z, \tau > T}] = \\
&= \exp\left(\frac{T\theta^2}{2}\right) \left[N\left(\frac{b(T) - T\theta}{\sqrt{T}}\right) - N\left(\frac{z - T\theta}{\sqrt{T}}\right) \right. \\
&\quad \left. + e^{2b_2(\theta - b_1)} \left(N\left(\frac{z - T\theta - 2b_2}{\sqrt{T}}\right) - N\left(\frac{b(T) - T\theta - 2b_2}{\sqrt{T}}\right) \right) \right].
\end{aligned} \tag{A7}$$

Proof. See theorem 3.2 from Zaeovski (2020c). □

Proposition A.4. If the functions $a(\cdot)$ and $b(\cdot)$ are linear, then the first exit probabilities are

$$\begin{aligned}
 P_1^l(T; a_1, a_2, b_1, b_2) &:= \mathbb{P}((\tau_1 \quad \tau_2) < T, \tau_1 < \tau_2) \\
 &= N\left(\frac{a_1 T + a_2}{\sqrt{T}}\right) \\
 &\quad + \sum_{j=1}^{\infty} \left\{ \begin{aligned} &e^{-2[-ja_2 + (j-1)b_2][-ja_1 + (j-1)b_1]} N\left(\frac{-a_1 T - 2(j-1)b_2 + (2j-1)a_2}{\sqrt{T}}\right) \\ &-e^{-2[j^2(a_1 a_2 + b_1 b_2) - j(j-1)a_2 b_1 - j(j+1)b_2 a_1]} N\left(\frac{-a_1 T - 2jb_2 + (2j-1)a_2}{\sqrt{T}}\right) \\ &-e^{-2[-(j-1)a_2 + jb_2][-(j-1)a_1 + jb_1]} N\left(\frac{a_1 T - 2jb_2 + (2j-1)a_2}{\sqrt{T}}\right) \\ &+e^{-2[j^2(a_1 a_2 + b_1 b_2) - j(j-1)b_2 a_1 - j(j+1)a_2 b_1]} N\left(\frac{a_1 T + (2j+1)a_2 - 2jb_2}{\sqrt{T}}\right) \end{aligned} \right\} \\
 P_2^l(T; a_1, a_2, b_1, b_2) &:= \mathbb{P}((\tau_1 \quad \tau_2) < T, \tau_2 < \tau_1)
 \end{aligned} \tag{A8}$$

$$\begin{aligned}
 &= 1 - N\left(\frac{b_1 T + b_2}{\sqrt{T}}\right) \\
 &\quad + \sum_{j=1}^{\infty} \left\{ \begin{aligned} &e^{-2[jb_2 - (j-1)a_2][jb_1 - (j-1)a_1]} N\left(\frac{b_1 T + 2(j-1)a_2 - (2j-1)b_2}{\sqrt{T}}\right) \\ &-e^{-2[j^2(b_1 b_2 + a_1 a_2) - j(j-1)b_2 a_1 - j(j+1)a_2 b_1]} N\left(\frac{b_1 T + 2ja_2 - (2j-1)b_2}{\sqrt{T}}\right) \\ &-e^{-2[(j-1)b_2 - ja_2][(j-1)b_1 - ja_1]} N\left(\frac{-b_1 T + 2ja_2 - (2j-1)b_2}{\sqrt{T}}\right) \\ &+e^{-2[j^2(b_1 b_2 + a_1 a_2) - j(j-1)a_2 b_1 - j(j+1)b_2 a_1]} N\left(\frac{-b_1 T - (2j+1)b_2 + 2ja_2}{\sqrt{T}}\right) \end{aligned} \right\}.
 \end{aligned}$$

Proof. See theorem 4.3 from Anderson (1960). □

Proposition A.5. We have for $\xi > 0$

$$\begin{aligned}
 L_1(t, \xi; a_1, a_2, b_1, b_2) &= \mathbb{E}[e^{-\xi \tau} I_{\tau \leq T, \tau = \tau_1}] \\
 &= e^{a_2(\sqrt{a_1^2 + 2\xi} - a_1)} P_1^l\left(T; \sqrt{a_1^2 + 2\xi}, a_2, b_1 + \sqrt{a_1^2 + 2\xi} - a_1, b_2\right) \\
 L_2(t, \xi; a_1, a_2, b_1, b_2) &= \mathbb{E}[e^{-\xi \tau} I_{\tau \leq T, \tau = \tau_2}] \\
 &= e^{b_2(\sqrt{b_1^2 + 2\xi} - b_1)} P_2^l\left(T; a_1 + \sqrt{b_1^2 + 2\xi} - b_1, a_2, \sqrt{b_1^2 + 2\xi}, b_2\right),
 \end{aligned} \tag{A9}$$

where the functions $P_1^l(\cdot)$ and $P_2^l(\cdot)$ are given by Eqs. (A8).

Proof. See theorem 3.1 from Zaeviski (2021a). □

Proposition A.6. *If the functions $a(t)$ and $b(t)$ are piecewise linear, then*

$$\begin{aligned} & \mathbb{E} \left[e^{-\xi \tau} I_{\tau \in (t_{m-1}, t_m)}, \tau = \tau_{1,2} \right] \\ &= \int_{\alpha_1, \dots, \alpha_{m-1}}^{\beta_1, \dots, \beta_{m-1}} \left(\prod_{i=1}^{m-1} \left(1 - \sum_{j=1}^{\infty} q_{ij}(x_{i-1}, x_i) \right) \frac{\exp \left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right)}{\sqrt{2\pi(t_i - t_{i-1})}} \right) \\ & \quad \left(e^{-\xi t_{m-1}} L_{1,2} \left(t_m - t_{m-1}, \xi; a_{m,1}, \alpha_{m-1} - x_{m-1}, b_{m,1}, \beta_{m-1} - x_{m-1} \right) \right) dx_1 \dots dx_{m-1}, \end{aligned} \quad (\text{A10})$$

where the functions $L_1(\cdot)$ and $L_2(\cdot)$ are given by Eqs. (A9). If the lower boundary vanishes, then Eqs. (A3) are actual and thus formula (A10) takes the form

$$\begin{aligned} & \mathbb{E} \left[e^{-\xi \tau} I_{\tau \in (t_{m-1}, t_m)}, \tau = \tau_1 \right] \\ &= \int_{\alpha_1, \dots, \alpha_{k-1}, -\infty}^{\beta_1, \dots, \beta_{m-1}} \left(\prod_{i=1}^{k-1} \left(1 - \sum_{j=1}^{\infty} q_{ij}(x_{i-1}, x_i) \right) \frac{\exp \left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right)}{\sqrt{2\pi(t_i - t_{i-1})}} \right) \\ & \quad \left(\prod_{i=k}^{m-1} \left(1 - \exp \left(-\frac{2(\alpha_{i-1} - x_{i-1})(\alpha_i - x_i)}{t_i - t_{i-1}} \right) \right) \frac{\exp \left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right)}{\sqrt{2\pi(t_i - t_{i-1})}} \right) \\ & \quad \left(e^{-\xi t_{m-1}} L(t_m - t_{m-1}, \xi; a_{m,1}, \alpha_{m-1} - x_{m-1}) \right) dx_1 \dots dx_{m-1}, \end{aligned} \quad (\text{A11})$$

where the function $L(\cdot)$ is given by Eq. (A6).

Proof. See theorems 4.1 and 4.2 from Zaeovski (2021a). $\square$

The four terms of $q_{1,j}$ from formula (A2) can be presented as an exponent of the linear functions $\lambda_{j,k}z + \eta_{j,k}$, $k = \{1, 2, 3, 4\}$, divided by T . Also, let $s_1 = s_3 = -1$ and $s_2 = s_4 = 1$.

Proposition A.7. *If the functions $a(t)$ and $b(t)$ are linear and $a(T) < z < b(T)$, then*

$$\begin{aligned} & V(\xi, z, T, a_1, a_2, b_1, b_2) = \mathbb{E} \left[e^{\xi B_T} I_{\tau > T, B_T < z} \right] \\ &= \exp \left(\frac{\xi^2 T}{2} \right) \left\{ l \left(N \left(\frac{z - \xi T}{\sqrt{T}} \right) - N \left(\frac{a(T) - \xi T}{\sqrt{T}} \right) \right) + \right. \\ & \quad \left. + \sum_{j=1}^{\infty} \sum_{k=1}^4 \left[s_k \exp \left(\lambda_{j,k} \xi + \frac{\lambda_{j,k}^2 + 2\eta_{j,k}}{2T} \right) \times \right. \right. \\ & \quad \left. \left. \left(N \left(\frac{z - (\xi T + \lambda_{j,k})}{\sqrt{T}} \right) - N \left(\frac{a(T) - (\xi T + \lambda_{j,k})}{\sqrt{T}} \right) \right) \right] \right\}. \end{aligned} \quad (\text{A12})$$

If we have piecewise linear boundaries, then Eq. (A12) turns into

$$\begin{aligned} & \mathbb{E} \left[e^{\xi B_T} I_{B_T < z, \tau > T} \right] \\ &= \int_{\alpha_1, \dots, \alpha_{n-1}}^{\beta_1, \dots, \beta_{n-1}} \left(\prod_{i=1}^{n-1} \left(1 - \sum_{j=1}^{\infty} q_{i,j}(x_{i-1}, x_i) \right) \frac{\left(\exp \left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right) \right)}{\sqrt{2\pi(t_i - t_{i-1})}} \right) \\ & \quad e^{\xi x_{n-1}} V \left(\xi, z - x_{n-1}, t_n - t_{n-1}; a_{n,1}, \alpha_{n-1} - x_{n-1}, b_{n,1}, \beta_{n-1} - x_{n-1} \right) dx_1 \dots dx_{n-1}, \end{aligned} \quad (\text{A13})$$

where the function $V(\cdot)$ is given by Eq. (A12). If the last k values of the lower boundary vanish, then Eq. (A13) turns into

$$\begin{aligned} & \mathbb{E} \left[e^{\xi B_T} I_{B_T < z, \tau > T} \right] \\ &= \int_{\alpha_1, \dots, \alpha_{k-1}, -\infty}^{\beta_1, \dots, \beta_{n-1}} \left(\prod_{i=1}^{k-1} \left(1 - \sum_{j=1}^{\infty} q_j(x_{i-1}, x_i; i) \right) \frac{\left(\exp \left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right) \right)}{\sqrt{2\pi(t_i - t_{i-1})}} \right) \\ & \quad \prod_{i=k}^{n-1} \left(1 - \exp \left(-\frac{2(\beta_{i-1} - x_{i-1})(\beta_i - x_i)}{t_i - t_{i-1}} \right) \right) \frac{\left(\exp \left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \right) \right)}{\sqrt{2\pi(t_i - t_{i-1})}} \\ & \quad e^{\xi x_{n-1}} V(\xi, z - x_{n-1}, t_n - t_{n-1}; b_{1,n-1}, \beta_{n-1} - x_{n-1}) dx_1 \dots dx_{n-1}. \end{aligned} \quad (\text{A14})$$

The function $V(\cdot)$ is given by formula (A7).

Proof. See theorems 3.2, 4.1, and 4.2 from Zaevski (2021a). □

Appendix B. Crank-Nicolson finite difference scheme

$$\begin{aligned} V_t &= \frac{V(m-1, n) - V(m, n)}{\Delta t} \\ V &= \frac{V(m-1, n) + V(m, n)}{2} \\ V_x &= \frac{V(m-1, n) - V(m-1, n-1) + V(m, n) - V(m, n-1)}{2\Delta x} \\ V_{xx} &= \frac{V(m-1, n+1) - 2V(m-1, n) + V(m-1, n-1)}{2(\Delta x)^2} \\ & \quad + \frac{V(m, n+1) - 2V(m, n) + V(m, n-1)}{2(\Delta x)^2}. \end{aligned} \quad (\text{B1})$$

$$\begin{aligned}
0 = & \frac{V(m-1, n) - V(m, n)}{\Delta t} + \\
& + \frac{1}{2} rx_n \frac{V(m-1, n) - V(m-1, n-1) + V(m, n) - V(m, n-1)}{\Delta x} \\
& + \frac{1}{4} \sigma^2 x_n^2 \left(\frac{V(m-1, n+1) - 2V(m-1, n) + V(m-1, n-1)}{(\Delta x)^2} \right. \\
& \left. + \frac{V(m, n+1) - 2V(m, n) + V(m, n-1)}{(\Delta x)^2} \right) \\
& - \frac{1}{2} r (V(m-1, n) + V(m, n)).
\end{aligned} \tag{B2}$$

- If $n = l_m + 1$, then

$$\begin{aligned}
& V(m, n) \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
& - V(m, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
& = V(m-1, n-1) \left(-\frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
& + V(m-1, n) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2} r \right) \\
& + V(m-1, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
& - V(m, l_m) \left(\frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right).
\end{aligned} \tag{B3}$$

- If $l_m + 1 < n < L_m - 1$, then

$$\begin{aligned}
& V(m, n-1) \left(\frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
& + V(m, n) \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
& - V(m, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
& = V(m-1, n-1) \left(-\frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
& + V(m-1, n) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2} r \right) \\
& + V(m-1, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2}.
\end{aligned} \tag{B4}$$

- If $n = L_m - 1$, then

$$\begin{aligned}
& V(m, n-1) \left(\frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
& + V(m, n) \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
& = V(m-1, n-1) \left(-\frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
& + V(m-1, n) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2} r \right) \\
& + V(m-1, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
& + V(m, L_m) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2}.
\end{aligned} \tag{B5}$$

Let us discuss briefly the arising system of linear equations. For a fixed t -node m , the unknown variables are $V(m, l_m + 1), \dots, V(m, L_m - 1)$ – their total number is $\eta = L_m - l_m - 1$. Thus the linear system can be written as $Ax = b$, where A is an $\eta \times \eta$ -square matrix and b is an $\eta \times 1$ -vector. For $i, j \in \{1, 2, \dots, \eta\}$, their non-zero elements are as follows:

1. If $i = 1$, then for $n = l_m + 1$ we have

$$\begin{aligned}
A_{11} &= \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
A_{12} &= -\frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
b_1 &= V(m-1, n-1) \left(-\frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
&+ V(m-1, n) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2} r \right) \\
&+ V(m-1, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} - V(m, l_m) \left(\frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right).
\end{aligned} \tag{B6}$$

2. If $1 < i < \eta$, then for $n = l_m + i$

$$\begin{aligned}
A_{i,i-1} &= \left(\frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
A_{i,i} &= \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
A_{i,i+1} &= -\frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \\
b_i &= V(m-1, n-1) \left(-\frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
&+ V(m-1, n) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2} r \right) + V(m-1, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2}.
\end{aligned} \tag{B7}$$

3. If $i = \eta$, then $n = L_m - 1$ and

$$\begin{aligned}
 A_{\eta, \eta-1} &= \left(\frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
 A_{\eta, \eta} &= \left(\frac{1}{\Delta t} - \frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} + \frac{1}{2} r \right) \\
 b_\eta &= V(m-1, n-1) \left(-\frac{1}{2} \frac{rx_n}{\Delta x} + \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} \right) \\
 &\quad + V(m-1, n) \left(\frac{1}{\Delta t} + \frac{1}{2} \frac{rx_n}{\Delta x} - \frac{1}{2} \frac{\sigma^2 x_n^2}{(\Delta x)^2} - \frac{1}{2} r \right) \\
 &\quad + V(m-1, n+1) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2} + V(m, L_m) \frac{1}{4} \frac{\sigma^2 x_n^2}{(\Delta x)^2}.
 \end{aligned} \tag{B8}$$

Therefore, the matrix A can be written as

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & \dots & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 & \dots & 0 \\ 0 & A_{32} & A_{33} & A_{34} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & A_{\eta-1, \eta-2} & A_{\eta-1, \eta-1} & A_{\eta-1, \eta} \\ 0 & 0 & \dots & 0 & 0 & A_{\eta, \eta-1} & A_{\eta, \eta} \end{bmatrix} \tag{B9}$$

Thus the system is well-defined and we derive the unknown values of $V(m, l_m + 1), \dots, V(m, L_m - 1)$ solving it. We do this using the platform MATLAB and its function `linsolve` based on the LU factorization with partial pivoting.